MASONRY DOMES: A STUDY ON PROPORTION AND SIMILARITY

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SUMMARY

A study of the old master builder's rules for structural design of arches, vaults and domes, reveals a persistent presence of "proportional rules", rules that produce structures geometrically similar. The square-cube law, however, demonstrates that in structures subject to their own weight, stresses grow linearly with size, invalidating these rules from an elastic point of view. A more detailed examination taking into account the problem of stability, i.e. the position of lines or surfaces of thrust, shows that the condition of sufficient stability in masonry structures is what causes an overal geometry for the structure. This excess of dimensions makes that, in fact, the square-cube law begins to apply only to very large spans. The size at which elastic design begins depends on the form of the structure, but for traditional forms, it clearly includes the dimensions of all historical architecture. The rigorous theoretical proof of this argument was implicit in Rankine's theorem of parallel projection as applied to masonry structures. In the present discussion the methods and concepts of Dimensional Analysis have been applied.

1. INTRODUCTION: GALILEO AND THE PRINCIPLE OF SIMILITUDE.

The origin of this work is research work in progress on the types of structures and the kind of structural formulae used in Spain for the construction of vaults and domes between 1500 and 1800[1]. In the course of this work it has become evident that architects and master masons relied on 'structural formulae' for structural design.

A great majority of these formulae are 'proportional', that is to say, they produce 'similar' forms in a geometrical sense; they give, for example, the depth of the buttress for an arch depending on its curve of intrados but regardless of its size. In other words, they implicitly believe in the existence of a 'law of similitude': a valid structural form continues to be correct independently of its size.

However Galileo demonstrated the impossibility of the existence of this kind of principle[2]. The reasoning is bright and clear: in structures supporting as the main load their own weight, as for example animals and masonry buildings, the dead load rises as the cube of the linear dimensions while the section of the structural members rise as the square; the tensions rise, therefore, linearly with the size . Figure.1 (a) shows Galileo's illustration of the effect of changes of size on the bone of an animal. Galileo's argument has achieved the rank of law, the 'square-cube law', in structural design. It has determined the attitude of engineers and architects to the effects of scaling in structural design, and of building and civil engineering historians towards the traditional proportional rules[3].

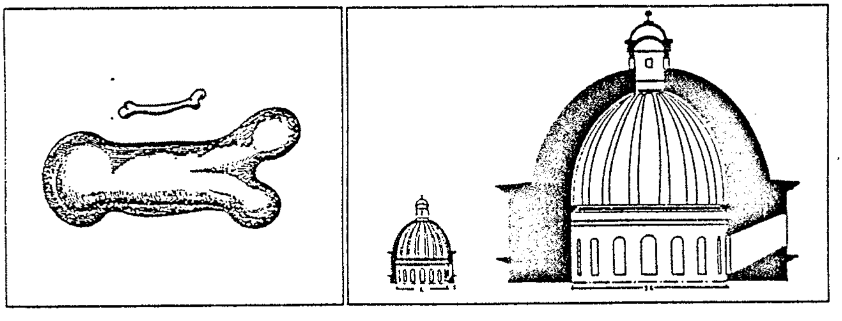


Figure.1 (a) (b)

Notwithstanding the truth and clarity of Galileo's proof, the traditional masonry structures appear to be quite similar, independently of their size. In Figure.1 (b) we have applied the square cube law of Galileo to the dome of the church of Saint Biagio in Montepulciano, Italy, in the same way as Galileo did for the bones of animals. The dome has an interior diameter of 14 meters; if we multiply this diameter by three we would obtain a dome with a diameter of 42 m. We have drawn to scale this dome with the dimensions resulting of the square cube law.

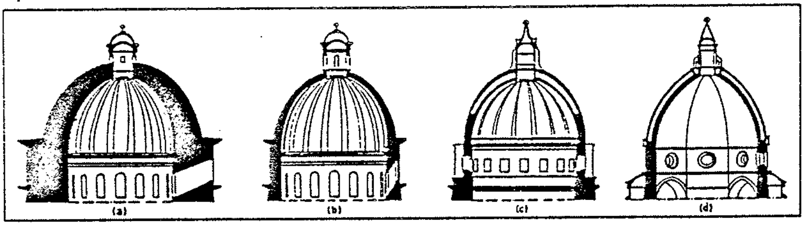


Figure.2

In Figure. 2 we compare the proportions of a dome of the same size obtained by the similitude principle, (b) with the dome obtained applying the square-cube law, (a), and with two domes which are 42 m of diameter: Saint Peter in Rome, (c), and Santa Maria del Fiore in Florence, (d).[4] Although the three domes are not exactly of the same form nor type of construction, clearley the 'law of similitude' functions much better than the square-cube law.

The same kind of comparison could be made with other existing masonry structures such as Gothic cathedrals, byzantine or roman domes, etc., with the same results. In the case of masonry towers and chimneys it appears that in fact an 'inverse law' functions, and towers are thinner as they grow in altitude (see Figure.3)[5].

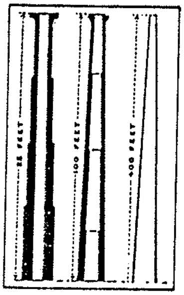


Figure.3

1.1 The effects of scaling up and down: the Principle of Similitude.

As we said before, Galileo was the first to study the effects of changes of size in structures. This kind of approach remained almost forgotten for approximately two hundred years[6]. At the end of the nineteenth century this subject began to arouse the interest of certain scientists, engineers and biologists, trying to solve by the use of models very complex physical problems, or to study the effects of size on animal and plant morphology. In the end these studies and the work of theoretical physicists and mathematicians on the homogeneity of physical equations, led to the founding of a new discipline 'Dimensional Analysis'[7].

We are concerned here with a branch of this discipline, 'the principle of similitude'. It consists on applying the laws of dimensional analysis to compare two geometrically similar systems; in Lord Rayleigh's words "the influence of scale upon dynamical and physical phenomena"[8]. Model theory belongs to this branch of dimensional analysis.

The kind of reasoning associated with the principle of similitude is extremely powerful and permits a very quick extraction of information about a phenomenon. Citing, again, Lord Rayleigh, "It often happens that simple reasoning founded upon this principle tell us nearly all that is to be learned from even a successful mathematical investigation"[9].

1.2 Similar structures

The main fields of application of the principle of similitude are fluid mechanics, heat and matter transfer,... and, in general, any discipline were experiments with models are necessary. In structures of buildings has been applied mostly in this last aspect, therefore mostly destined to particular cases and not to extract general conclusions of design.

However the use of this principle permits the formulation of very general conclusions as to the design of structures, or to infer new solutions from existing ones. The contributions in this sense are very scarce.

The first exponent of this approach is Rankine who in his 'Manual of Applied Mechanics' he derived a 'method of parallel projection' which allowed him to draw conclusions between a certain structure and its projection on a given plane. He applied his method to hanging cables, polygonal frames and masonry structures. Later we will discuss the consequences of the application of the method on masonry arches and vaults. Besides, throughout his work there are frequent remarks as to the effect of variations of size. The works of Thomson[10] and Barr[11] should also be cited in this sense.

In the rest of the article a study of the stability of masonry arches, vaults and domes is made on the effects of scaling up and down, employing the methods and reasoning derived from the similitude principle, and making some reflections on the possibility of deducing some kind of safe proportional rules. These considerations should throw new light on the structural design and historic interpretation of masonry structures[12].

2. MASONRY ARCHES.

We will consider first the case of masonry arches, and shall use it later in the study of domes. Suppose we have a masonry arch, subject to its own weight, and defined in such a way that we can scale it up and down, for example by a certain middle line and a law of variation of thickness. If this arch is to be 'safe' certain conditions must be fulfilled with respect to strength and elasticity, and to stability.

The first one, which corresponds to the application of elastic theory, imposes that the material should not reach a certain level of stress considered 'unsafe' or non admissible.

The second condition, which corresponds to the application of limit analysis, imposes certain restrictions to the position of the line of thrust. Due to the incapacity of masonry to resist tensile stresses, an arch will be stable, for a given system of loads, if at least one line of thrust can be found lying anywhere within the masonry[13]. This produces a 'lower limit' for the arch thickness, i.e. a 'limit form', for the given system of loads. (See Figure. 4(a).) We can now fix this limit form of the arch employing some 'form factors' or non-dimensional parameters. As this is the limit state of equilibrium we reach a certain degree of security, in fact we make the arch more stable, providing that the line of thrust departs sufficiently from the lines which limit the arch. We can apply then a 'geometrical factor of safety' stating that the line of thrust will pass always within a certain fraction of the thickness, say 1/3 or 1/4[14]. (See Figure. 4(c).)

The two conditions mentioned above must be fulfilled at the same time in any arch. We will study first the case of dead load only, and after that we will explore the effect of live loads.

2.1 Dead load

In the case of dead load only, the position of the line of thrust is determined by the geometrical shape of the structure, and, therefore, its limit form is independent of the scale. The application of the geometrical factor of safety gives rise to similar forms. However, with respect to strength the increase of thickness is a linear function of the linear dimensions.

It's best to see the effect of both factors by means of an example. Suppose we have an arch of constant thickness and a certain form and that we know its limit form. In this case it can be represented by only one form factor, the ratio thickness/span (t/s) which we will call 'slenderness' of the arch.

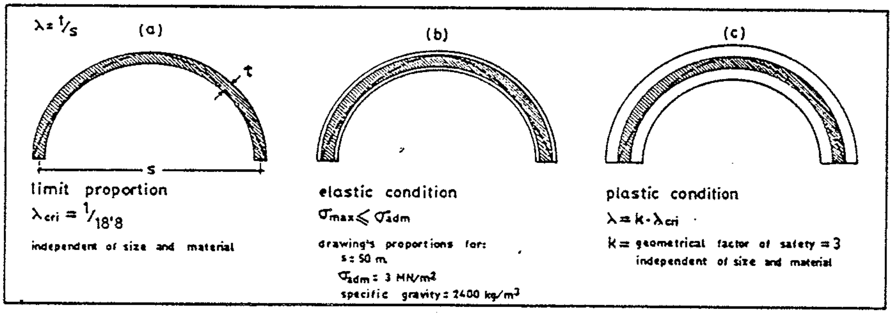
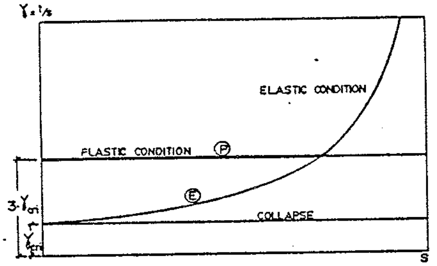


Figure.4

We take this 'limit arch' as a point of departure for the design. For a certain value of the span the elastic condition imposes an increase of thickness just to reach the value of the admissible stress in the point of maximum stress (Figure.4(b)). If we plot a curve relating the slenderness of the arch (t/s) to the span (s) we obtain curve E in Figure.5, showing an increase in thickness as the span rises.

In plastic or limit condition, the geometrical safety factor mentioned above, produces an horizontal straight line P (Figure.5). The domain of safe design is above these two lines. The point of intersection represents the dimension at which the plastic design ceases to be the critical.



This point marks the range of dimensions for the validity of proportional rules for the design of arches. Its position depends on the form of the arch and on the specific gravity and admissible stress of the masonry.

Figure.5

In Figure.6 we have plotted the same curves for two of the most common masonry arches: a semicircular arch standing alone, and with filling up to the level of the keystone. The limit slenderness for the first is 1/18.8 and for the second 1/44, applying a geometrical factor of 3 we obtain values of approximately 1/6 and 1/14. These horizontal lines intersect the corresponding curves for elastic design at values of the span of 82 and 106 meters respectively (admissible stress = 3 MN/m^2; specific gravity of masonry = 2.2 g/cm^3; specific gravity of filling = 1.8 g/cm^3) [15]. As the tensions increase linearly with the dimensions, these results can be easily extrapolated to other admissible tensions.

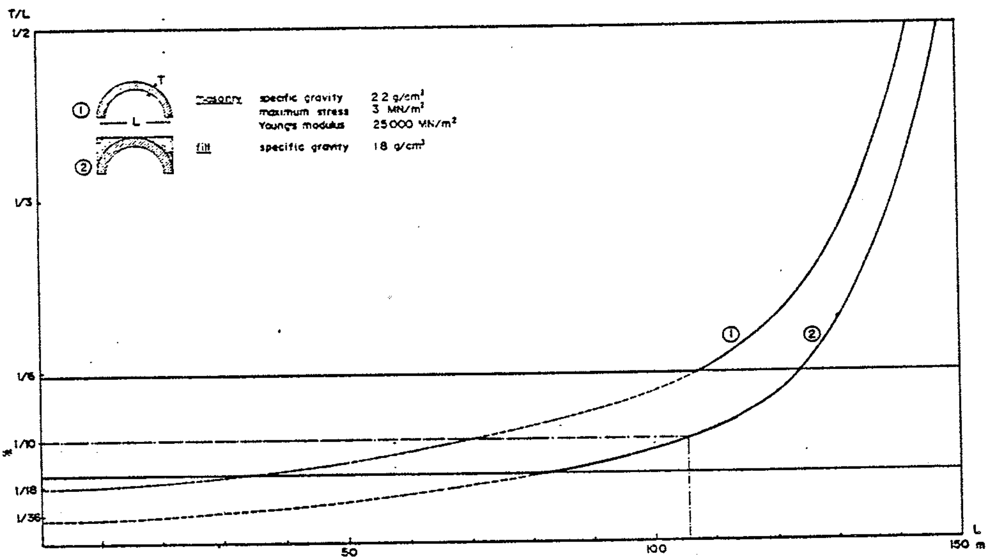


Figure.6

As it is easily seen the critical point is well over the dimensions of any masonry structure of this form ever constructed, as all the greatest semi-circular arches ever built are under the 60 meters of span[16].

The same type of considerations can be made for other type of traditional arches (ogival, basket, elliptical,...), that are found in historical architecture. In general, the form imposed by stability produces a low stress level for a range of dimensions that comprise easily all historical architecture.

Therefore, for structures subject only to dead load, or where it is the most significative load, as for example the masonry vaults of churches and cathedrals, the use of proportional rules, i.e. of non-dimensional form parameters, is a rational and safe design method.

The easiest way to specify this form parameters for a system of similar structures is to give a series of fractions. That is precisely what traditional master builders did. The gothic master builders expressed in that way the ratio thickness/span for the ribs of the cross vaults[17] and this type of rules for arch and vault design has been used until this century (they can be found in many engineering and building Handbooks).

2.2 Live loads

Now, we will consider the action of a point load P that can be placed at any point along the span of the arch. In this case, we can obtain for a certain value and position of P and a certain value of the span a limit slenderness for the arch, and we will call P the critical load. If we change the position of P we obtain a series of values of the limit slenderness, the maximum of which represents the global limit slenderness for the critical load P and span s.

This slenderness will be greater than the values corresponding to the dead load, and therefore we should increase correspondingly the thickness of the arch, applying to it the geometrical factor.

¿How does the critical load P change in similar structures? To study the variation of the critical load P for a given arch in relation to the span we can use the method of dimensions with great benefit. The variables that enter in the problem are: the point load P, the span of the arch, s, and the specific gravity μ of the masonry (supposing arch and filling of equal material). Dimensional analysis gives the following equation[18]:

$$P = \mu\, s^2\, \Phi(w_i, w_j, ..)$$

where Φ is a function of w_i, w_j,... form factors derived from the geometry of the arch and the position of the loads. We have supposed that P is a force acting per unit of length perpendicular to the plane of the arch. If P is just a point load it will be proportional to s^3.

In two similar arches of different material:

$$\frac{P}{P'} = \frac{\mu\, s^2}{\mu'\, s'^2}$$

if they are of the same material:

$$\frac{P}{P'} = \frac{s^2}{s'^2}$$

In the first case the critical point loads are in proportion to the specific gravities and the square of linear dimensions, i.e. to the weights of the arches (as we are considering a section between two parallel planes). In the second case the critical point loads are in proportion to the square of the ratio of similarity.

Although dimensional analysis says nothing of the form of the function Φ the result is important because it easily permits the extrapolation of a particular case. It is simple to obtain the value of Φ for a given case:

$$\Phi(w_i, w_j, \ldots) = \frac{P_1}{\mu_1 s_1^2}$$

and then we can obtain the value for any similar structure of any material (μ) and size (s)[19].

It is clear from the form of the equation that any pair of values (Φ,P), (P,s) or (Φ,s) define the remaining one. Say, for any given dimension and value of P there exist a critical slenderness. To see the variation in a certain case, in Figure.7 we have plot the results for a semi-circular arch of constant thickness with the load P applied in the keystone.

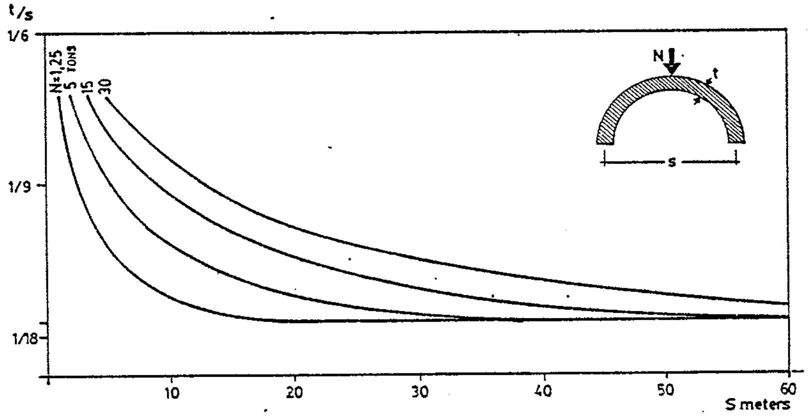


Figure.7

We can now make some considerations of the effects of the changes of size on similar structures with a live load.

For similar structures of the same material, to have the same degree of stability, P must vary with the inverse of the square of the ratio of similarity.

However, in normal practice, this is not the case and the value of P is fixed in function of the placement and use of the structure. We can consider then P as a constant. In this case, it is easy to see that if we have a stable arch any similar arch of greater size, subject to the same system of loads, would be 'more stable' as the line of thrust would deviate much less from its center line.

In fact, from the point of view of stability, the greater the arch the thinner it could be made. This kind of 'inverse law' applies also in the case of masonry towers subject to wind (see Figure.3 above). The only check would be to assure that the level of stresses does not rise above the admissible level of the material. Actually, the empirical rules used in the design of bridges gave all for similar arches a decreasing thickness in the crown. In Figure.8 we have represented the well known rules of Perronet, Croizette-Desnoyers and Dupuit[20]. We have also represented the rule given by Martinez de Aranda[21], a spanish architect of the sixteenth century, in his manual of steorotomy. The rule applies explicitly only to arches of a given range of dimensions.

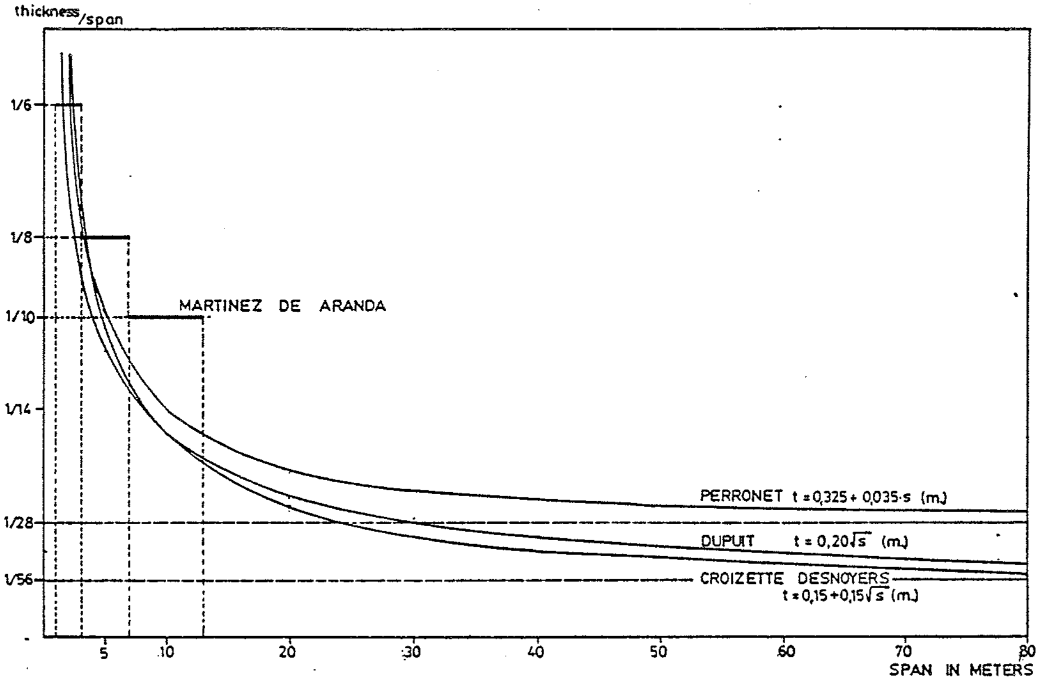


Figure.8

On the contrary, an arch of lesser dimensions would need a thorough verification of its stability. However, if we know the critical load of the original arch, we can calculate immediately the critical load of the new arch. In the case that the two point loads expected were not the same sometimes this verification would suffice. In this instance no stress check is necessary.

This seems to invalidate the proportional rules for the case where pointed load are significant as is the case of bridges, as for every span we can find a unique solution. In fact they will be valid if they represent an upper bound for the slenderness for the most unfavorable load. This is the situation in most cases with the slenderness of Roman and Renaissance bridges, comprised typically between 1/8 and 1/12 of the span.

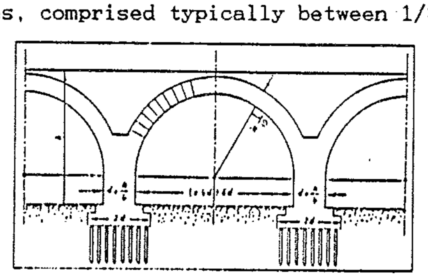


Figure.9

Figure.9 represents Alberti's rule for bridges as represented by Straub[22]. Perhaps the more rational approach, and one that could permit to benefit from the use of similar forms is that of Martínez de Aranda, also used by Gauthey, defining simple ratios for certain intervals of dimensions.

2.3 Limits of size

The limit of size for a masonry arch depends on the admissible tension and specific gravity of the material, and on the form of the arch. Applying again dimensional analysis we obtain:

$$\sigma_{max} = \Phi(w_1, w_2, \ldots)\, \mu\, s$$

where σ_{max} is the maximum stress, μ is the specific gravity of the material, s is the span and $\Phi(w_1, w_2, \ldots)$ is a function of the form factors. The value of Φ could be used to measure the efficiency of a gravity arch.

The question as to what size a masonry arch can be built have worried engineers ever since the beginning of structural analysis. Leading engineers as Perronet[23], Dupuit, Sejourné[24] and Freyssinet[25], dedicated much thought to this problem. To serve as a point of reference, Table.1[26] lists the greatest masonry arches in the world, with an indication of the maximum stress levels, where available.

It is remarkable the Bridge over the river Adda in Trezzo, Italy, with a clear span of 72 m. Another bridge not included in the table should be mentioned: Leonardo projected a bridge for Sultan Bajezid II over the Golden Horn in Istanbul, with an span of 240 m. What remains of this project are a few sketches but the form of the bridge is clerarly defined and permitted Stüssi[27] an analysis of its stability and strength. He concluded that the project was feasible, the bridge being stable and presenting a maximum stress of 9 MN/m² at the haunches.

GREATEST MASONRY BRIDGES	SPAN (m)	MAX. STRESS (MN/m²)
Bridge of Walnut-Lane (USA, 1906-8)	71	2.7
Bridge over l'Adda in Trezzo (Italia, XIV cent.)	72	-
Bridge of Montanges (France, 1909-9)	80	5.0
Bridge of Salcano (Austria, 1904-5)	85	5.1
Bridge of Luxemburgo (Lux. 1899-1903)	85	4.8
Bridge over Rocky River (USA, 1908-10)	85	4.4
Bridge of Plauen (Germany, 1903-5)	90	6.9
Bridge of Villeneuve (France, 1914-1919)	93	5.7
Bridge of Bernand (France, project. 1913)	165	8.1

Table.1

The greatest modern projected bridge in unreinforced concrete; the 'Viaduc' of Bernand, was projected by Freyssinet with an span of 190 m, was never constructed due to the outbreak of First World War.

This huge bridges seems to establish the limit for the span of masonry bridges. However, Freyssinet believed firmly on the possibility of erecting arch bridges of more than 1 Km of span; he fixed the limit around 2 Km basing it on the economy of the centering dreams[28]. As far as we know, the greatest arch is that of Gladesville Bridge, in Australia, made in reinforced concrete, with a clear span of 1000 ft. (305 m.) presenting a maximum stress of 14 MN/m²[29].

2.4 Rankine's Theorem of Parallel Projection

Until now we have spoken of similar structures. The similarity we have spoken of is a restricted case of a more general geometrical relation between figures: parallel projection. As it has been said before, this general case have been studied by Rankine, who enunciated a 'method of parallel projection'.

Two figures are 'parallel projections' one of the other if between them exists a relation such that to each point in one their corresponds another point in the other, and, that to each system of two equal, parallel lines corresponds in the other another system of two equal, parallel lines.

Rankine represented the parallel projection by its mathematical expression. Given a figure defined by its coordinates respect to some axes x,y,z, rectangular or oblique, a second figure defined respect to other axis x', y', z', is a parallel projection of the first if between any two corresponding points, their coordinates verify the following equations:

$$\frac{x'}{x} = a, \ \frac{y'}{y} = b, \ \frac{z'}{z} = c,$$

where a, b, c are constants. Later we will give a more simple geometrical interpretation for plane figures.

Parallel projections of plane and spatial figures have a series of geometrical properties[30]. These allow to draw conclusions as to the variation of lengths, surfaces and volumes, and its partitions and centers of gravity. It can be applied also to the transformation of systems of forces in equilibrium. In this last aspect Rankine enunciated the following theorem: "...if a balanced system of forces acting through any system of points be represented by a system of lines, then any parallel projection of that system of lines will represent a balanced system of forces."[31]

Rankine applied it to study the effects of change and form on the equilibrium of articulated frames, hanging cables, linear arches and masonry structures. His studies on the transformation of frames led to the formulation, by Maxwell, of the theorem of 'reciprocal figures'[32].

With respect to masonry arches he enunciated a theorem on the stability of the 'transformation of constructions formed of blocks'. This theorem states that if a construction formed of blocks and subject to the action of a system of loads represented by lines accomplish the condition of stability, any parallel projection will have the same degree of stability, provided that the loads are the parallel projection of the original loads.
This means that the relative position of the line of thrust respect the lines of extrados and intrados will be the same. That represents the possibility of obtaining, given an stable arch for a system of loads, an infinite number of stable arches[33].

We have enunciated before the mathematical definition of a parallel projection. For a plane figure it is easy to find a simple geometrical interpretation: given a figure referred to two axis x,y which forms an angle ø, any parallel projection can be reduced to the combination of three basic transformations: extension or reduction in the dirction of any one of the axes, and variation of the angle ø between them. In Figure.10 we have represented this basic transformations and some combinations of them. As the original arch is stable, any of the projections has the same degree of stability.

The theorem is very powerful and allows the rapid solution of some practical problems as to the adequate the form of the arch to a given relation of height to span or to produce an asymmetrical arch with its springings at different levels. For example, the three arches represented in Figure 11 produce the same horizontal thrust. However arch A contains half the material as arch B, and arch C double. Stresses are in an inverse proportion and are double in A and half in C, taking as a reference arch B.

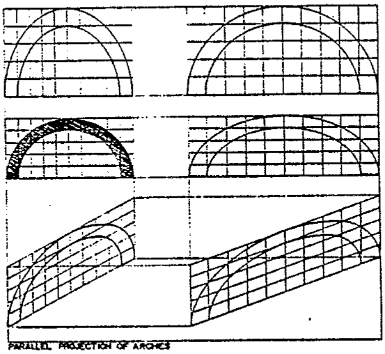


Figure.10

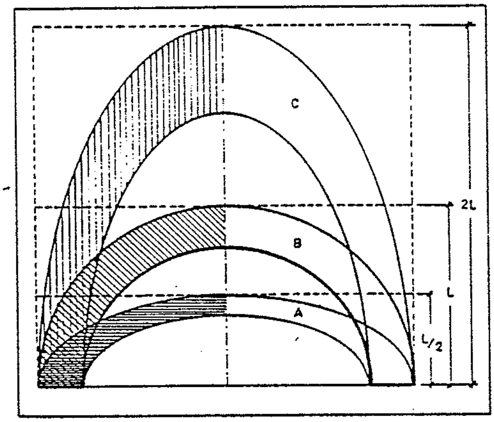


Figure.11

Another example is shown in Figure.12, where all the arches with buttresses presents the same degree of stability.

Of course tensions would change and a little calculation would have to be made if the tensions on the original arch are near the admissible limit or if we produce a large change of size. Mathematical formulae could be derived to obtain the increment of stresses, and indeed Rankine gives one[34], but it would be more simple and free of errors to obtain it graphically.

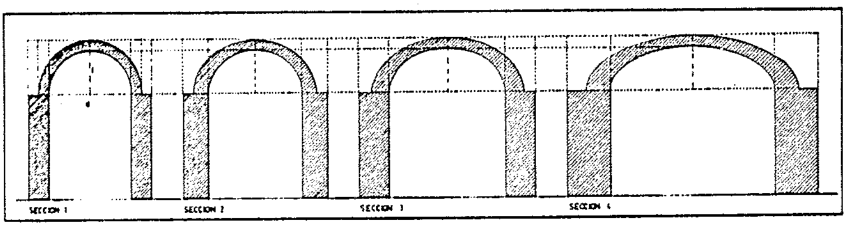


Figure.12

3. MASONRY DOMES

The most important difference between a voussoir arch and a voussoir dome lies in that while in the arch we only consider the stresses on two radial sections, in the dome the voussoirs are subject to stresses on all four faces. This makes domes more stable than arches and permits, for example, to remove the keystones and open an 'oculus' without the collapse of the dome.

If all stresses were compressive, there would be no problem at all, and they are so in the upper part of domes. But to a certain angle 'hoop' tensile stresses appears that masonry are not prepared to resist. The angle at which this occurs depends on the form of the dome and on the system of loads applied to it. In a semicircular dome of constant thickness tensile stresses appear at an angle $\alpha = 51.8°$ from the top. The lantern, or even an 'oculus', makes the zone of tensile stresses grow towards the top.

Due to this phenomenon and to the incapacity of masonry to resist tensile stresses, the problem of the masonry dome can be reduced to that of the masonry arch: the dome below the point of zero stress splits into separate fragments, lune shaped, which act as a series of radiating arches. This has been the traditional approach[35] to the study of masonry domes and has been recently revived by Heyman[36]

Therefore a dome may burst apart in radiating lines near the bottom and nonetheless be stable provided that the abutment is sufficient. Old master builders have been aware of this fact and, for example, in the dome of Santa Sophia, as Dunn[37] has rightly pointed out, the architect acknowledging this effect created windows round the base, in such a way that this part of the construction is formed by forty-four separate radiating arches[38].

3.1 Limit proportions of domes

The same observations made for arches are valid for domes. A dome, therefore should present, to be stable, a certain limit form. As the distribution of the charges is more favorable in the dome they can be made much thinner than arches.

In the case of a hemi-spherical dome of constant thickness Heyman[39] has obtained for the 'slenderness' of the dome, thickness/span, a value of 1/47.6[40], that is to say we can make the dome approximately 2.5 times thinner than the semi-circular arch. It is advantageous to make the dome pointed and for a dome generated by the revolution of an equilateral arch the limit slenderness would be 1/73[41].

As in the case of arches, to obtain a certain degree of security we apply a geometrical factor of safety, typically comprised between 2 and 3.

This 'valid' form is independent of the size and masonry domes maintain it - the form derived from stability considerations - for an interval of dimensions greater than in the case of arches. If in the case of arches there are some that are near the limit, existing masonry domes are very far away from this limit, as we shall see later.

3.2 Pointed loads

The effect of pointed loads could be disregarded because the function of this type of structure as roofs excluded the apparition of very strong point loads. When they appear, as in the case of heavy lanterns, they form part of the permanent dead load and should be taken from the beginning in the study of the equilibrium of the structure.

The equation deduced using dimensional analysis as to the influence of pointed loads in similar arches applies fully, and the critical load rises with the cube of linear dimensions. This, of course, is what happens with lanterns.

3.3 Rankine's Theorem

Rankine's theorem of parallel projection applies fully to the case of domes or of any spatial masonry structure. The practical application of this theorem to the case of domes is even more useful due to the much more complicated analysis of domes.

Consider for example the case of a masonry dome of ellipsoidal form with three different principal axes. The study of the stability and stress level on this structure would involve painstakingly long calculations as the arches generated by the dome in the moment of collapse would split are all different for a quarter of the base.

We can obtain an immediate answer applying a parallel projection of the stable hemi-spherical dome whose proportions we know (see above) just multiplying each of the coordinates, x, y, z, by the desired factor to obtain a dome of the desired proportions. The variations of the stresses could be obtained, again, analytically using Rankine's equation or graphically for the points where the higher rise is expected.

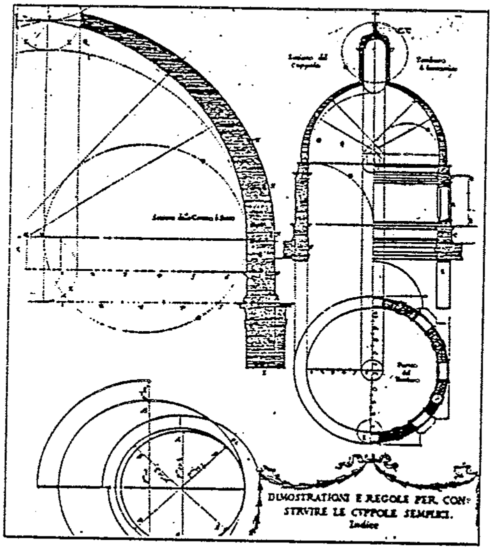
3.4 The dome of Saint Biagio and Fontana's rule

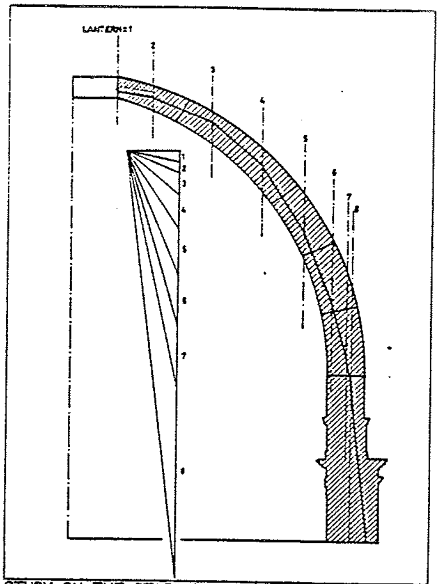
We begin this article with a comment about the dome of Saint Biagio and a comparison with the domes of Saint Peter and Santa Maria del Fiore. We have chosen the dome of Saint Biagio for two reasons; the first because is almost an exact fraction (1/3) of the greatest masonry domes and that permitted us to establish a simple relationship; the second is that it corresponds approximately to the proportions of a geometrical rule which has some diffusion in Renaissance Italy. The rule was published by Carlo Fontana in his book on Saint Peter[42] and is reproduced on Figure.13 (a). We have made a graphical analysis of the stability of this type of dome, base on the above exposed hypothesis of limit analysis of masonry domes (see Figure.13 (b)). As it can be seen, the design is very satisfactory as the line of thrust is always contained within the middle third of the section. As for the tensions, a simple application of dimensional analysis show that:

$$\sigma_{max} = \Phi(w_1, w_2, \ldots)\ \mu\ s$$

where Φ is a function of the form factors, μ is the specific gravity of the material and s is a linear dimension of the dome (we can take for example the span s at the base of the dome).

The calculated value of Φ for a dome of this form is approximately of 1.28[43]. The corresponding σ_{max} for Saint Biagio, s = 14 m., μ = 2 g/cm^3, is of 0.34 MN/m². A similar dome three times greater would present a σ_{max} of 1.02 MN/m² which is not much even for a brick of medium quality. Now we will compare this value with the actual σ_{max} in the other two domes with which we have made the comparison. The proportions of the dome in Figure.2 (a) would be attained for an span of 370 m., for an admissible stress of 3 MN/m².





STUDY ON THE STABILITY OF FONTANA'S DOME

Figure.13 (a) (b)

In the case of Saint Peter, μ being equal, the value of σ_{max} is, following Gottgetreu's estimates[44], of 1.06 MN/m². The value of Φ is very similar, 1.29. The position of the line of thrust is not as favorable as in Saint Biagio - it passes at 1/5 of the thickness at the base - but this is, perhaps, compensated by the reduction of weight in the superior part by means of the double shell[45].

Parsons[46] gave for Santa Maria del Fiore a maximum stress of 2.5 MN/m² but this is for the most unfavorable hypothesis of supposing all the weight concentrated on the ribs. If we suppose the weight distributed following the line of thrust for the cloister vault solution then we would obtain 1.02 MN/m², Φ being equal to 1.22. The inconvenience of using the octagonal form is compensated with the more favorable position of the thrust line, passing neatly within the middle third, and the use of the double shell.

The levels of stresses resulting from the previous analysis are all quite moderate. Of course the greatest stresses are to be found not in the domes (surface elements) but in the pillars (linear elements), but even in that case stresses does not approach a dangerous level for good masonry. For example, in the main pillars of Saint Peter, perhaps the greatest masonry building, the stress is of 1.7 MN/m²[47].

Considering that a good masonry could present an admissible stress one order of magnitude higher, 10-20 MN/m², it is a fact that masonry domes and buildings have not even approach their limits of possible size (see Table.2 for a list of the greatest masonry domes). For example multiplying the dimensions of Saint Peter by three we would obtain a maximum stress on the pillars of 5.1 MN/m^2 which is not excessive; the problem would be the total volume of the resulting fabric; it is a problem of human scale and purpose not of the material[48][49].

GREATEST MASONRY DOMES	SPAN (m)
Pantheon (Rome, I cent.)	43
S. Sophia (Istanbul, VI cent.)	33
S. Maria del Fiore (Florence, XV cent.)	42
S. Pietro (Rome, XVI cent.)	42
Gol Gomuz (Bijapur, India, XVII cent.)	42
S. Paul (London, XVIII cent.)	33
S. Francisco el Grande (Madrid, XVIII cent.)	33
Mosta church (Mosta, Malta, XIX cent.)	38
S. Carlo (Milan, XIX cent.)	32

Table.2

4. CONCLUSIONS: ON THE VALIDITY OF TRADITIONAL PROPORTIONAL RULES

The stability of masonry structures subject chiefly to its own weight imposes certain overall dimensions, in fact some geometrical form. This form supposes, from an elastic point of view, an over-dimensioning of the structure. As a result stress levels in masonry structures of traditional sizes (say, spans of less than 60 m.) are low, and the condition of stability is more restrictive.

This conduces to certain 'valid forms' for masonry arches, vaults and domes - in fact even for buildings. The traditional geometrical rules provided a means to 'fix' this safe proportion of masonry structures and are, consequently a rational and valid form for the structural design of masonry structures, within the normal range of dimensions cited above. These considerations have been expressed several times by Heyman[50], but they have received little attention by building and civil engineering historians.

In fact, this approach justifies the tremendous success of traditional master builders that lies, as Gordon has stated, in that "...the nature of the design problem in large masonry buildings is peculiarly adapted to the limitations of the pre-scientific mind"[51]. The possibility of using geometrical rules, of relying on models, and above all, the use of previous buildings as 'full-scale' models, has undoubtedly played an essential role. It permits to explain the impressive success of structures such as the Pantheon and Saint Sophia, almost doubling in size any structure ever constructed before.

Besides, the possibility of making just a 'visual checking' of the stability of an arch or dome cannot be overemphasized, above all in a profession where drawing is the most important means of expression and transmission of knowledge[52]. In fact, most drawings of arches and domes that are found in the old treatises represent 'good forms'.

Today masonry structures have no great diffusion and in any case, beams and frames have replaced arches in almost every instance. However there are two fields, besides the historical field already mentioned, where the previous remarks are pertinent. These are the restoration of old masonry constructions, and, the problem of building in the Third World. In this last instance perhaps the geometrical rules and proportions of masonry could be easily understood and applied by a non-specialized work force (being, besides, masonry the cheapest material).

NOTES

[1]. S. Huerta Fernández "Structural design of arches, vaults and domes in Spain: 1500-1800". Doctoral Dissertation in progress under the direction of Professor R. Aroca Hernández-Ros.

[2]. Galileo Galilei *Discorsi e Dimostrazioni Matematiche intorno à due nuove scienze*. Leiden: 1688. pp. 233 y ss.

[3]. The general attitude of engineering historians pointed to the impossibility of deducing valid rules due to the ignorance of the science of static. See: Parsons, W.P. Engineers and Engineering in the Renaissance, Cambridge, The MIT Press, 1965 (reprint of 1939 edition), pp. 481; Mainstone, R.J. Developments on Structural Form, Harmondsworth, Penguin, 1983, pp. 284; Dorn, H.I. The Art of Building and the Science of Mechanics: An Study of the Union of Theory and Practice in the Early History of Structural Analysis in England. Ph.D. Princeton University, 1970, pag. 50-51; Benvenuto, E. La Scienze della Costruzione ed il suo sviluppo storico. Firenze, Sansoni,1982, pp.234-235.

[4]. The dimensions and form from the cited domes has been taken from Durm, J. Die Baukunst der Renaissance in Italien. Leipzig: 1914.

[5]. See: Barr, A. Address on the Application of the Science of Mechanics to Engineering Science. London: The Institution of Civil Engineers, 1899, from which Figure.3 has been taken, pag. 11.

[6]. After Galileo, Borelli, De Motu Animalium, 1685, applied the same approach to demonstrate that a man would never be able to fly and to explain why small animals leap higher. See: Thompson, D'Arcy Wentworth. Galileo and the principle of Similitude. Nature. Vol. 95, 1915, pp. 426-427. Isaac Newton in his Philosophiae Naturalis Principia Mathematica, 1687, Book II, Section VII, Prop. XXXII, Theorem XXVI, applied the principle for the first time to a dynamical problem, the movement of two similar bodies. See: Greenhill, G. Mechanical Similitude. Mathematical Gazette, Vol.8, 1916, pp. 229-233.

[7]. Palacios, J. Análisis Dimensional, Madrid, 1964, pp. 9-17.

[8]. Lord Rayleigh Nature, Vol.95, 1915, pp. 202-203.

[9]. Nature, Vol. 95, 1915, pp. 202-203.

[10]. Thomson, J.J. Comparisons of Similar Structures as to Elasticity, Strength, and Stability. Transactions of the Institution of Engineers and Shipbuilders of Scotland, 1875.

[11]. Barr, A. Address on the Application..., op.cit. above, and Barr, A. Comparisons of Similar Structures and Machines. Transactions of the

Institution of Engineers and Shipbuilders of Scotland, Vol. 42, 1899, pp. 322 and ff. We have not been able to consult this last work.

[12]. The basic correctness of the traditional structural formulae in respect to the principles of plastic design of masonry structures has been mentioned several times by J. Heyman. See for example: Beauvais Cathedral. Transactions of the Newcomen Society, Vol.40, 1967/68, pp. 15-35; On the Rubber Vaults of the Middle Ages. Gazette des Beaux-Arts, Vol. 71, 1968, pp. 177-188. See also: Gordon, J.E. *Structures* (Harmondsworth: 1977), Chap.9 "Walls, arches and dams", pp. 171-197.

[13]. This is the first, or 'lower bound', theorem, or limit analysis applied to arches. See: Kooharian, A. Limit Analysis of Voussoir Segmental Arches. Proceedings of the American Concrete Institute, Vol.49; Heyman, J. The Safety of Masonry Arches. International Journal of Mechanical Sciences, Vol.11, 1969, pp. 363-385; and Parland, H. Basic principles of the structural mechanics of masonry: a historical review. International Journal of Masonry Construction, Vol.2, n°2, 1982, pp. 48-58.

[14]. See: Heyman, J. The Masonry Arch. Chichester: 1982.

[15]. The calculations are made under the following assumptions: a) the line of thrust does not suffer a significative deviation due to the increasing of the thickness. b) the diagram of stresses forms a rectangle due to the plastification of the material.

[16]. For an inventary of all the great vaults over 40 m. of span, see: Sejourné, P. Grandes Voûtes. Bourges: 1913-1916, 6 vols.

[17]. See for example, Kubler, G. A late gothic computation of rib vault thrusts. Gazette des Beaux-Arts, Vol.26, 1944, pp.434-439, and Shelby, L. and Mark, R. Late gothic structural design in the Instructions of Lorenz Lechler. Architectura, Vol.9, 1979, pp. 113-131.

[18]. We have followed the method given in J. Palacios, Análisis Dimensional, Madrid, 1964, pp. 78 and ff. and 113-114.

[19]. Heyman has obtained the value of the Φ function for the case of the point load P placed at a quarter of the span. See: Heyman, J. The estimation of the strength of masonry arches. Proceedings of the Institution of Civil Engineers, Part 2, Vol. 69, 1980, pp. 921-937, and, The Masonry Arch. Chichester, U.K., Ellis Horwood, 1982, pp. 72-78.

[20]. See: Croizette-Desnoyers, Ph. Cours de Construction des Ponts. Paris, Dunod, 1885, Vol.2, pp. 1-30; Dupuit, J. Traité de l'équilibre des voûtes et de la construction des ponts en maçonnerie. Paris, Dunod, 1870, pp. 169-197. For an 'elastic interpretation' see: Davidesco, M. Examen critique des formules employées pour déterminer l'épaisseur a la clef des voûtes en maçonnerie. Annales des Ponts et Chaussées, 1906, pp.247-253.

[21]. Martínez de Aranda, G. Cerramientos y trazas de montea. Ms. Biblioteca del Servicio Histórico Militar, Madrid. (Undated, approximately end of XVIth century). Fac. edition Madrid, CEHOPU, 1986.

[22]. Straub, H. A History of Civil Engineering. London, Leonard Hill, 1952. p.90.

[23]. Perronet, J.R. Mémoire sur la recherche que l'on pourroit employer pour construire de grandes Arches de pierre ... Paris, Imprimairie Nationale, 1793.

[24]. Sejourné, P. Quelques réflexions pratiques sur les voûtes a grande portée. Annales des Ponts et Chaussées, 1886, pp. 497-502.

[25]. Freyssinet, E. Perfectionnement dans la construction des grandes voûtes. Le Génie Civil, Vol. 58, 1921, pp. 97-102, 124-128, 146-150; and, Les Ponts en béton armé de trés grande portée. Mémoires de la Societé des Ingenieurs Civiles de France, 1930, pp. 376-379.

[26]. The information has been taken, chiefly from Sejourné, P. Grandes Voûtes, op. cit. above.

[27]. Stüssi, F. Leonardo da Vincis Entwurf einer Brücke über das Goldene Horn. Schweizerische Bauzeitung, Vol.71, 1953, pp. 113-116.

[28]. See: Freyssinet, op. cit. above and Fernández Ordóñez, F. Eugene Freyssinet. Barcelona, 2C Ediciones, 1978, pp. 377-378.

[29]. Baxter, J.W.; Gee, A.F. and James, H.B. Gladesville Bridge. Proceedings of the Institution of Civil Engineers, Vol.30, 1965, pp. 489-530.

[30]. See: Rankine, W.J.M. Applied mechanics. Glasgow, 1864, chapter IV, Parallel projections in static.

[31]. Ibidem.

[32]. See: Timoshenko, S.P. History of Strength of Materials. New York, Dover, 1983 (reprint of 1953 ed.), pp. 197-208, and Charlton, T.M. A History of the Theory of Structures in the Nineteenth Century. Cambridge, U.K., Cambridge University Press, 1982, pp. 58-66, 73-93.

[33]. The only limit to this arise from the dependence of the form of the line of thrust on the family of section planes. A good exposition of this dependence is in: Dupuit, J. Traité de l'équilibre des voûtes et de la construction des ponts en maçonnerie. Paris, Dunod, 1870. For the complete mathematical theory of thrust lines see: Milankovitch, M. Theorie der Drückkurven. Zeitschrift für Mathematik und Physik, Vol. 55, 1907, pp. 1-27.

[34]. Manual of Applied Mechanics, op. cit., paragraph 235-A.

[35]. Although well known since antiquity, this hypothesis was employed for the first time to estimate the stability of the dome of Saint Peter in the two studies make by experts called by the Pope: Le Seur, T.; Jacquier, F.; Boscovich, R.G. Parere di tre mattematici sopra i danni, che si sono trovati nella cupola di S. Pietro. Roma, 1743, and Poleni, G. Memorie istoriche della Gran Cupola del Tempio Vaticano. Padova, 1748. After that it constitued the standard method of analysis throughout the XIX century.

[36]. See: Heyman, J. On Shell Solutions of Masonry Domes. International Journal of Solids and Structures, Vol.3, 1967, pp. 227-241, and, Heyman, J. Equilibrium of Shell Structures. Oxford, U.K., Oxford University Press, 1977, pp. 106-116.

[37]. Dunn, W. The Principles of Dome Construction. Architectural Review, Vol. 23, 1908, pp. 63-73 and 108-112.

[38]. This practice seems to have been used also in roman times. Sometimes these arches does not even reach the top of the dome. See: Torres Balbás, L. Bóvedas romanas sobre arcos de resalto. Archivo Español de Arqueología, Vol. 64, 1946, pp. 173-28.

[39]. Heyman, J. Equilibrium of Shell Structures. Op. cit. above.

[40]. This value almost coincides with the value of 1/43.4 calculated by Sir Edmund Beckett, who has been, as far as we know, to study the limit proportions of domes in his article: On the Mathematical Theory of Domes. Memoirs of the Royal Institute of British Architects, 1871 Feb, pp. 81-115. A summary, made by the same author, of the most important results appear in the article 'Dome' in the Encyclopaedia Britannica, 9th ed., Edinburgh, U.K., 1875-1888, Vol. VII, pp. 347-348.

[41]. Sir Edmund Beckett, op. cit. above.

[42]. Fontana, C. Il Tempio Vaticano e sua origine. Roma, 1694. Cap. XXIV, Regole per le Cupole Semplici..., pp. 361-367.

[43]. The value of Φ for a hemi-spherical dome is 1. The pointed form makes the global volume greater and tensions rise. However as we have seen the form is more stable.

[44]. The calculation has been made taking as a point of departure the investigation made by Gottgetreu. He was concerned only with the position of the line of thrust. Knowing it, the above mentioned stress is matter of a simple arithmetical operation involving the total weight of the dome and its surface at the base. See: Gottgetreu, R. Lehrbuch der Hochbaukonstruktionen. Vol.I, pp. 254-260, Tafel. XXIX, 'Stabilitätuntersuchung der Peterskuppel zu Rom'.

[45]. Surprisingly the analysis made by Durand-Claye of the dome projected by Bramante, which is massive and inspired in that of the Pantheon gave a σ_{max} of 0.88 MN/m², with Φ = 1.07. See: Durand-Claye, A. Vérification de la stabilité des voûtes et des arcs. Application aux voûtes sphériques. Annales des Ponts et Chaussées, 1880, pp. 416-440, planches 14-16.

[46]. Parsons, W.B. Engineers and Engineering in the Renaissance. Cambridge, Mass., The MIT Press, 1976 (reprint of 1939 ed.), pp. 587-600.

[47]. Navier, L.M.N.H. Resumé des Leçons données à l'Ecole des Ponts et Chaussées sur l'Application de la Mécanique à l'Etablissement des Constructions et des Machines. Bruselas, 1839, p. 102.

[48]. The actual volume of Saint Peter is of 107.988 m^3. A lineal increase of the dimensions by three will multiply this by 27 and that will make 2,915.676 m^3, which is more than the volume of Cheops' pyramid (2,592.100 m^3). Saint Peter's volume after the measures and calculations made by Fontana (see op. cit. above); Cheops' pyramid volume from its overall dimensions.

[49]. Mäkelt. Der Bau gro er gemauerten Kuppelgewölbe. Zentralblatt der Bauverwaltung, Vol. 62, 1942, pp. 409-417, proposes the construction of a dome of 100 m of span made of brickwork.

[50]. See note 12 above.

[51]. See note 12 above. The citation from Op.cit. p.171.

[52]. Ferguson has shown the importance of drawings on the transmission of old machines design. See: Ferguson, E.S. The Mind's Eye: Non-verbal Thougth in Technology. Science, Vol.197, 1977, pp. 827-836.

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